Stability Results of Nonlinear Rieman Stiltes integraty equation.

Bekchanov Sardor Ergash ogli

(Urgench state university)

Bekchanovsardor94@gmail.com

Fayzullayev Shoxzodbek Maqsud ogli

(Urgench Ranch technology university)

@fayzshoxzod@gmail.com

Abstract: *In this paper, the stability of a class of nonlinear integro-differential equation is investigated and analyzed. By defining a suitable Lyapunov functional we establish necessary and sufficient condition -for the stability of the zero solution. Our results extends known results in the literature.*

Keywords: *Integro-differential equation, Lyapunov functional, Nonlinear, Stability.*

1. Introduction

Volterra integro-differential equations have wide applications in biology, ecology, medicine, physics and other scientific areas and thus has been extensively studied. The equilibrium or the steady state of a linear or nonlinear equation can either be stable or unstable. The steady state is called a stable system if after been disturbed by some physical phenomenon returns to its uniform state of rest or its normal position. When a system tends to a new position after a slight displacement, such equilibrium is called unstable equilibrium.

The origin of stability in science and engineering can be track down to the work of Aristotle and Archimedes (Magnus 1959). Alexander Lyapunov was the first to define the notion of stable system in Mathematical form in 1892, in his book on “the general problem of stability”.

The stability theorem for motion studied by A.M Lyapunov has proven to be highly useful and applicable in the field of science and engineering. The notion of stability is studied in the literature under three classes, namely; Bounded input and bounded output (BIBO), Zero-input stability and Input-state Stability. Over the years, Lyapunov method for the stability of integro-differential equation have been proposed by different researcher (Stamove and Stomov (2001, 2013), Tunc (2016), Tunc and Sizar (2017) Vanulailai and Nakagiri (2003), Carabollo et al. (2007), Segeev (2007)).

In particular Stamova and Stomov (2001) worked on the stability of the zero state solution of impulsive function differential equation. They applied the Lyapunov-Razumikhin method and Piecewise continuous function to check the behavioral solution of equation. Vanualailai and Nakagiri (2003) established stability of systems of Volterra integro-differential equation. They used a known form of Lyapunov functional to establish the stability condition for the system. Carabollo et al. (2007) worked on construction of lyapunov functionals to check and investigate the stability for hereditary system. Cemil (2016) studied certain nonlinear Volterra integro differential equations with delay. He established stability and boundedness condition of the solution by defining a suitable Lyapunov functional used to prove the result.

Sergeev (2007) establishes the stability of the solutions of a class of integro-differential equations of Volterra type whose nonlinear term is assumed to be holomorphic function of variables and possible some integral form in a small neighborhood of zero. Stability in Lyapunov’s sense of single zero root and of pair of pure imaginary roots for the unperturbed equation is analyzed by relying on functional in integral form represented by Frechet series.

2. Preliminaries

Our aim in this paper is to use a suitable Lyapunov functional and determine necessary and sufficient condition for the stability of the zero solution of the nonlinear integro – differential equation of Volterra type defined by

$$y'(t) = B(t)g(y(t)) + \int_0^1 G(t,s,y(s)) \qquad (2.1)$$

Where $y \in R$, the funktions G is continuous in (t,s,y) for $0 \le s \le t < \infty$, $B(t)$ continuous for $0 \le t < \infty$, $g(y(t))$ is continuous on $(-\infty, \infty)$ and

$$\int_0^t G(t,s,y(s))ds < \infty, \quad \int_0^t tG(t,s,y(s))ds < \infty \qquad (2.2)$$

We use following natation and basic information throughout this paper. For any $t_0 \ge 0$ and initial function $\varphi \in [t_0, t]$ let $y(t) = y(t, t_0, \varphi)$ denote the solution of eq. (2.1) on $[t_0, t)$ such that $y(t) = \varphi(t)$. Let $C([t_0, t_j])$ and $C([t_0, \infty])$ denote the continuous of real vaued functions on $[t_0, t_j)$ and $[t_0, \infty)$ respectively.

$$\text{For } \varphi \in C[t_0, 0], \|\varphi\| = \sup\{|y(t)| : 0 \le t \le t_0\}.$$

Theorem 2.1 [Driver (1962)]. If there exists a functional $V(t, \varphi(.))$, defined whenever $t \ge t_0 \ge 0$ and

$\varphi \in C([0, t], \mathbb{R}^n)$ such that

i. $V(t,0) \equiv o, V$ is continuous in t and locally Lipschitz in φ

ii. $V(t, \varphi(t)) \ge W(|\varphi(t)|), W : [0, \infty) \to [0, \infty)$ is a continuous function with $W(0) = 0, W(r) > 0$ if $r > 0$ and W is strictly increasing (positive definiteness), and

iii. $V(t, \varphi(.)) \le 0$ then the zero solution of eq. (2.1) is stable and

$$V(t, \varphi(.)) = V(t, \varphi(s)) : 0 \le s \le t$$

Is called a Lyapunov function of eq. (2.1)

3. Main result

Theorem 3.1 *If* $B(t)g(y(t)) < 0,\ G(t, s, y(s)) > 0$ *and*

$$B(t)g(y(t)) + \int_0^t G(t, s, y(s))ds \neq 0 \qquad (3.1)$$

Then the statements below are equivalent

i. The solution of eq. (2.1) tends to zero.

ii. $B(t)g(y(t)) + \int_0^t G(t, s, y(s))ds < -\xi,\ \xi > 0$

iii. Every solution of (2.1) is a Lebesgue integrable function with respect to the vector space $\mathbb{R}^n$.

Proof We shall adopt the method of Lakshmikantan (1995) to show that (iii) ⇒ (i), (i) ⇒ (ii) and (ii) ⇒ (iii).

Given that $q \in L'(R_+)$, the zero solution $y = 0$ of eq. (2.1) is uniformly stable if and only if the two positive functions $m(t)$ *and* $n(t)$ are uniformly bounded on R_+ it is uniformly asymptotically stable if and only if it is uniformly stable and both $m(t)$ *and* $n(t)$ tend to zero as $t \to \infty$.

We are going to show that (i) ⇒ (ii) and (ii) ⇒ (iii) and we are done.

Let $y(t, t_0, 1) > 1$ be any solution of (2.1) with initial function $q(t) = 0$ on the interval $[0, t_0]$. We claim that $y(t) = y(t, t_0, 1) > 0$ on $[0, t_0]$ if not, there exists a $t_1 > t_0$ with $y(t_1) = 1$. Hence $y'(t_1) < 0$, thus it follows from (i) that

$$\begin{aligned} y'(t) = B(t)g(y(t)) + \int_0^{t_1} G(t, s, y(s))ds > \\ B(t)g(y(t)) + \int_0^{t_1} G(t, s, y(s))ds \geq 0 \end{aligned} \qquad (3.2)$$

This is contradiction. Thus, (i) ⇒ (ii), then we are left to prove that (ii) ⇒ (iii).

Choosing a functional candidates

$$V(t, s, g(y(t))) = y(t) + \int_0^t \int_0^\infty G(\beta, s, y(s))\, d\beta\, g(y(s))ds \qquad (3.3)$$

Then for all $y \neq 0$, assuming $y(t)$ is a solution of (2.1), differentiating (3.3) along the solution of (2.1), we have

$$
\begin{aligned}
V'\left(t,s,g(y(t))\right) &\leq B(t)(y(t)) + \int_0^t G(t,s,y(s))\, g(y(s))\, ds \\
&\quad + \int_0^t G(t,s,y(s))\, d\tau\, g(y(s))\, ds - \int_0^t G(t,s,y(s))\, g(y(s))\, ds \\
&= B(t)(y(t)) + \int_0^\infty G(t,s,y(s))\, d\tau\, g(y(s)) \\
&= g(y(t)) \left[B(t) + \int_0^\infty G(t,s,y(s))\, d\tau \right]
\end{aligned}
$$

$$V'\left(t,s,g(y(t))\right) = -g(y(t))\,\xi$$

Where

$$\xi > 0, g(y(t)) > 0, \text{ then } \int_0^\infty g(y(s))\, ds < \infty \quad \text{and}$$

$$\int_0^\infty G(\tau,s,y(s))\, d\tau < \infty$$

Thus the solution of eq. (2.1) is Lebesgue integrable having satisfy the conditions of a Lebesgue integral

Conclusion

The behavior of integro-differential equation is frequently described by the construction of lyapunov functional. The method of lyapunov functional construction has a wide range of application in investigating the stability of functional differential equation, difference equation with continous or discrete time etc. In this paper by constructing a suitable Lyapunov function we proved necessary and sufficient condition for the stability of the zero solution of a class of nonlinear integro-differential equation.

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